BLOW-UP IN THE PARABOLIC SCALAR CURVATURE EQUATION

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ABSTRACT. The *parabolic scalar curvature equation* is a reaction-diffusion type equation on an $(n-1)$-manifold Σ, the time variable of which shall be denoted by r. Given a function R on $[r_0, r_1) \times \Sigma$ and a family of metrics $\gamma(r)$ on Σ, when the coefficients of this equation are appropriately defined in terms of γ and R, positive solutions give metrics of prescribed scalar curvature R on $[r_0, r_1) \times \Sigma$ in the form
$$g = u^2 dr^2 + r^2 \gamma.$$
If the area element of $r^2 \gamma$ is expanding for increasing r, then the equation is, in fact, uniformly parabolic, and the basic existence problem is to take positive initial data at some $r = r_0$ and solve for u on the maximal interval of existence, which above was implicitly assumed to be $I = [r_0, r_1)$; one often hopes that $r_1 = \infty$. However, the case of greatest physical interest, $R > 0$, often leads to blow up in finite time so that $r_1 < \infty$. It is the purpose of the present work to investigate the situation in which the blow-up nonetheless occurs in such a way that g is continuously extendible to $\bar{M} = [r_0, r_1] \times \Sigma$ as a manifold with totally geodesic outer boundary at $r = r_1$.

1. INTRODUCTION

Given a smooth family of Riemannian metrics $\gamma(r), r \in [r_0, \infty)$ on an $(n-1)$-manifold Σ, the *parabolic scalar curvature equation* refers to the equation

$$(1) \qquad \bar{H} r \frac{\partial u}{\partial r} = u^2 \Delta_\gamma u + Au - \frac{1}{2} \left(\bar{R} - r^2 R \right) u^3,$$

where $\bar{R}_r$ is the scalar curvature of $\gamma(r)$, the function R is "arbitrary", and the remaining terms in the coefficients are defined by

$$A = r \frac{\partial \bar{H}}{\partial r} - \bar{H} + \frac{1}{2} |\bar{\chi}|_\gamma^2 + \frac{1}{2} \bar{H}^2,$$

$$\bar{\chi}_{AB} = \gamma_{AB} + \frac{1}{2} r \frac{\partial \gamma_{AB}}{\partial r},$$

$$\bar{H} = \mathrm{tr}_\gamma \bar{\chi} = (n-1) + \frac{1}{2} r \frac{\partial \gamma_{AB}}{\partial r} \gamma^{AB};$$

A, B are used to denote components with respect to local coordinates θ^i on Σ. Positive solutions of Equation (1) on an interval $[r_0, r_1)$ give metrics g of prescribed scalar curvature R on $M = [r_0, r_1) \times \Sigma$ in the form

$$g = u^2 dr^2 + r^2 \gamma;$$

see [3] [15], [16], [13]. The function $\bar{H}$ and tensor $\bar{\chi}$ are closely related to the geometry of M. Indeed, with H, χ the mean and extrinsic curvatures of the hypersurfaces $\Sigma_r = \{r\} \times \Sigma$, one has

$$\bar{H} = ruH$$

$$\bar{\chi} = \frac{u}{r}\chi.$$

In the case that $f \equiv r^2 R/2 - \bar{R}/2$ is positive and bounded away from 0, it is easily established by using the maximum principle that solutions will not exist for all $r > 0$ but will in fact blow up for some finite value of r. It is the purpose of the present work to investigate the blow-up behavior in the case that Σ is compact.

The simplest case of blow-up, which we shall refer to as the trivial case, occurs under the assumption that f is fixed and positive, and u is constant on each Σ_r. Then $\bar{\chi} = \gamma$ and $\bar{H} = (n-1)$ so that Equation (1) is reduced to the ordinary differential equation

$$(n-1)r\frac{du}{dr} = \frac{(n-1)(n-2)}{2}u + fu^3.$$

For "initial" data $u(r_0) = u_0$, the solution of this problem is

$$u(r) = \frac{1}{\sqrt{c_0\left(\left(\frac{r_1}{r}\right)^{n-2} - 1\right)}},$$

where $r_1^{n-2} = \frac{(n-1)(n-2)}{2f_0}u_0^{-2}r_0^{n-2} + r_0^{n-2}$ and $c_0 = \frac{2f_0}{(n-1)(n-2)}$. Although the solution clearly blows up at $r = r_1$ the metric $g = u^2 dr^2 + r^2\gamma$ is defined up to and including r_1 as a C^∞ metric on a manifold with boundary, which is totally geodesic at $r = r_1$. This is seen by making the change of variables $\tilde{r} = \tilde{r}_0 + \int_{r_0}^{r_1} u\,dr$, which puts the metric in the form $g = d\tilde{r}^2 + r^2\gamma$. It is natural to ask: *more generally, when can we expect this behavior?* As a partial answer to this question, in this work the following theorem is proved:

Main Theorem A. *Let Σ be a compact $(n-1)$-manifold with a fixed metric γ. Let R be a C^∞ function on $[r_0, r_1 + \varepsilon)$, $\varepsilon > 0$ such that $r^2 R$ is non-decreasing. Let u be a solution of Equation (1) on $[r_0, r_1)$ such that*

$$\inf_{[r_0,r_1)\times\Sigma} u\sqrt{r_1 - r} \geq \mu > 0.$$

Then $\lim_{r\to r_1} u\sqrt{r_1 - r}$ exists and is a positive C^∞ function ω on Σ so that the metric

$$g = u^2 dr^2 + r^2\gamma = \frac{\omega^2}{r_1 - r}dr^2 + r^2\gamma$$

is extendable to $\bar{M} = [r_0, r_1] \times \Sigma$ in the sense that $g \in C^\infty(M) \cap C^0(\bar{M})$.

Thus, when f is positive and non-decreasing, one can assert that if u blows up everywhere on Σ at the blow-up time $r = r_1$, and the blow-up happens *at least* as fast as in the trivial case, then the solution blows up exactly at this rate, which after a change of variables allows the corresponding metric to be extended in the sense of C^0 to the boundary component $r = r_1$. Of course, it is relevant and highly important to ask whether or not there are actually any non-trivial examples of this. In fact, many such examples can be constructed from solutions of the curve shortening flow in the plane. These are discussed in a brief appendix. More

importantly, one has a stability result, which shows that when (Σ, γ) is the fixed round sphere then there is a neighborhood of non-trivial examples around the trivial one. The statement of this result appears at the end of the introduction, and the result is proved in Section 6 after the proof of the main theorem.

The proof of the main theorem proceeds after the observation that when γ is fixed in r some simple changes of variables transform Equation (1) into a more manageable form. To see this, note that when γ is fixed $\bar{\chi} = \gamma, \bar{H} = n - 1$ so that $A = (n-1)(n-2)/2$ and Equation (1) is

$$(n-1)r\partial_r u = u^2 \Delta u + \frac{(n-1)(n-2)}{2}u + fu^3,$$

where the subscript γ on the Laplacian has been dropped as will be done in the remainder. Then the function $\tilde{u} \equiv r^{1-\frac{n}{2}}u$ verifies

$$(n-1)r^{(3-n)}\frac{\partial \tilde{u}}{\partial r} = \tilde{u}^2 \Delta \tilde{u} + f\tilde{u}^3.$$

Thus, defining

$$t = \frac{r^{n-2}}{(n-1)(n-2)},$$

and regarding $\tilde{u} = \tilde{u}(p, t)$, our equation takes the much nicer form

$$(2) \qquad \frac{\partial \tilde{u}}{\partial t} = \tilde{u}^2 \Delta \tilde{u} + f\tilde{u}^3,$$

Note that this equation has the scaling property that if $\tilde{u}(p, t)$ is a solution then $\lambda \tilde{u}(p, \lambda^2 t)$ is also a solution; this will be used below to assume without loss of generality that the blow-up time occurs at $t = 1$. Note also that the order of the 'space' and 'time' variables has been switched from what it was previously to the more standard order for parabolic equations.

In the case that f is also fixed, examples of blow-up that occur exactly like the special case discussed above can now be be generated, in principle, by separation of variables: the function $\tilde{u} = v/\sqrt{t_1 - t}$ verifies Equation (2) provided $v \in C^\infty(\Sigma)$ is a positive solution of the *stationary equation*

$$(3) \qquad \Delta v + f_{t_1}v - \frac{1}{2v} = 0.$$

Following terminology as for the porous medium equation, solutions $v(p)/\sqrt{t_1 - t}$ generated in this way will be called *self-similar*. These solutions, if they exist, are very special. But, Main Theorem A asserts that in general if a solution blows up somewhere on Σ at the rate suggested by the self similar blow-up, then in fact it blows up almost exactly like a self-similar solution.

To prove that more generally blow-up is essentially self similar, one can follow the same procedure used to generate self similar solutions, with the generalization that the scaled function v is now allowed to depend on t. That is, defining $v = \sqrt{t_1 - t}\tilde{u}$, study the equation for v:

$$(t_1 - t)\frac{\partial v}{\partial t} + \frac{1}{2}v = v^2 \Delta v + fv^3$$

Assuming without loss of generality that the blow-up occurs at $t_1 = 1$, a final change of variables $t = 1 - e^{-\tau}$ yields

$$(4) \qquad \frac{\partial v}{\partial \tau} = v^2 \Delta v + fv^3 - \frac{1}{2}v,$$

and the blow-up behavior of the original equation can be dealt with by studying the behavior of v as $\tau \to \infty$. Specifically, the main theorem now follows from:

Main Theorem B. *Suppose that f is a C^∞ function on $[\tau_0, \infty) \times \Sigma$ such that $\|f\|_{C^k([\tau_0,\infty)\times\Sigma)}$ is bounded for each $k \in \mathbb{N}$. Suppose furthermore that*

$$\frac{\partial f}{\partial \tau} \geq 0. \tag{5}$$

Let v be a positive solution of Equation (4) on $[\tau_0, \infty)$ that satisfies

$$v \geq \mu \tag{6}$$

for some positive constant μ. Then there exists a positive solution $\omega \in C^\infty(\Sigma)$ of the stationary equation, Equation (3), such that $\lim_{\tau\to\infty} v = \omega$ in the sense of C^k for any $k \in \mathbb{N}$.

The proof of this theorem, in turn, results from the successive application of the following three theorems that will be proved in the remainder:

Theorem 1. *Any solution v of Equation (4) on an interval $[\tau_0, \infty)$ satisfying $v \geq \mu$ for some positive constant μ in addition satisfies $v \leq M$ for some constant $M \leq \infty$.*

Theorem 2. *Let v be a positive solution of Equation (4) on an interval $[\tau_0, \infty)$, which satisfies $\mu \leq v \leq M$ for some positive constants μ, M. Then there exists a sequence τ_i such that $v(\tau_i)$ converges uniformly to a positive C^∞ solution ω of the stationary equation.*

Solutions of Equation (3) will be referred to as *stationary states*. This theorem asserts that the ω-limit set of v is non-empty; it contains a stationary state. As a consequence, we see that if the hypotheses of the theorem are satisfied and in addition f is fixed then there is a self-similar solution $\omega/\sqrt{1-t}$.

Theorem 3. *Let v be a solution of (4) on $[\tau_0, \infty)$ satisfying $\mu \leq v \leq M$, and let ω be a positive C^∞ stationary state in the ω-limit set of v, where convergence is taken in the sense of C^0. Then ω is unique and $\lim_{\tau\to\infty} v(\tau) = \omega$, where the limit can be taken in the sense of $C^k(\Sigma)$ for any k.*

The outline of the paper is as follows:

Section 2 presents some basic pointwise inequalities that are fundamental for most of the bounds in the remainder of the paper. These inequalities are similar to inequalities derived for the porous medium equation, originally by Aronson-Bénilan [2]. The condition that γ be fixed is crucial.

Section 3 is devoted to the proof of Theorem 1. This is accomplished by proving a strong global Harnack inequality for v that shows that $\sup_\Sigma v(\tau, p)$ is bounded in terms of $\inf_\Sigma v(\tau + h, p)$ followed by a maximum principle argument that shows that $\inf_\Sigma v(p, \tau + h)$ is globally bounded from above.

Theorem 2 is proved in Section 4. This is done using techniques similar to those used by C. Cortazar, M. Pino, and M. Elgueta in [5], [6], [7] to study blow-up in the porous medium equation with source. The main tool is the functional

$$J(v) = \int_\Sigma \left(|\nabla v|_\gamma^2 - f v^2 + \log v \right) dV_\gamma,$$

which is non-increasing by virtue of Equation (4) and Condition (5). The bounds $\mu \leq v \leq M$ then show that $J(v)$ is bounded from below, which leads to the existence

of a sequence τ_n such that $v(\cdot, \tau_n)$ converges to a stationary state weakly in H^1 and strongly in L^1.

In Section 5 results of Leon Simon [12] are used to prove Theorem 3.

In Section 6, the promised stability result is proved, which we now state. First, note that in the case that γ is the fixed round metric and $f \equiv 1/2$, one has the blow-up solution $u = \sqrt{(n-1)(n-2)}/\sqrt{(r_1/r)^{n-2} - 1}$ with initial data $i_0 \equiv u|_{r_0} = \sqrt{(n-1)(n-2)}/\sqrt{(r_1/r_0)^{n-2} - 1}$, and which blows up at $r = r_1$.

Theorem 4. *Let (Σ, γ) be the fixed round sphere, and assume $f \equiv 1/2$. There exists a neighborhood B of i_0 in $C^{2,\alpha}(\mathbb{S}^{n-1})$ and a Lipschitz function $\tilde{r} : B \to \mathbb{R}$ such that given $u_0 \in B$, the solution of Equation (1) with initial data u_0 at r_0 is of the form*

$$u = \frac{\sqrt{(n-1)(n-2)}v}{\sqrt{\left(\frac{\tilde{r}(u_0)}{r}\right)^{n-2} - 1}},$$

where v is bounded and $\lim_{r \to r_1} v = 1$.

2. ARONSON-BÉNILAN INEQUALITIES

Let $\tilde{u}, v$ be solutions of Equations (2) and (4), respectively. The fundamental pointwise inequalities upon which the other crucial bounds depend are

$$(7) \qquad t\frac{\partial \tilde{u}}{\partial t} > -\frac{1}{2}\tilde{u},$$

$$(8) \qquad (1 - e^{-\tau})\frac{\partial v}{\partial \tau} > -\frac{1}{2}v,$$

and the integrated versions

$$(9) \qquad \tilde{u}(p, t_2) > \sqrt{\frac{t_1}{t_2}}\tilde{u}(p, t_1),$$

$$(10) \qquad v(p, \tau_2) > e^{-\frac{\tau_2 - \tau_1}{2}}\sqrt{\frac{1 - e^{-\tau_1}}{1 - e^{-\tau_2}}}v(p, \tau_1),$$

for $t_2 > t_1$ and $\tau_2 > \tau_1$. To get these, we need only assume that γ is fixed and $\partial f/\partial \tau \geq 0$.

These are proved by an Aronson-Bénilan type argument similar to that used for the porous medium equation [2]. To implement this here, we define $w = 1/\tilde{u}$ so that Equation (2) becomes

$$(11) \qquad \frac{\partial w}{\partial t} = -(\Delta + f)w^{-1}.$$

Defining now

$$z \equiv t\frac{\partial w}{\partial t} - \frac{1}{2}w = -t(\Delta + f)w^{-1} - \frac{1}{2}w,$$

one finds

$$(12) \qquad z' = -(\Delta + f)w^{-1} + t(\Delta + f)w^{-2}w' - f'tw^{-1} + \frac{1}{2}(\Delta + f)w^{-1},$$

where time differentiation has been denoted by a prime. It is now easily seen, using Condition (6), that z satisfies the linear parabolic differential inequality

$$(13) \qquad z' \leq (\Delta + f)w^{-2}z.$$

By the parabolic maximum principle, since z is negative initially, it must remain so. Whence

$$t\frac{\partial w}{\partial t} < \frac{1}{2}w,$$

and this inequality is equivalent to Inequalities (7) and (8).

3. Proof of Theorem 1

As indicated in the introduction, in this section Theorem 1 is proved by using the maximum principle to show that $\inf v$ must remain bounded. This establishes the result since, also in this section, we obtain a Harnack inequality that bounds $\sup v$ in terms of $\inf v$. The latter is contained in Proposition 8, whose proof is a direct consequence of the weak Harnack inequalities of the next three lemata, which are generalizations of results of Caffarelli and Friedman [4]. The first of these establishes a lower bound on $\inf v$ in terms of $\int_\Sigma v$, and the second and third, in turn, use $\int_\Sigma v$ to bound $\sup v$ from above.

Lemma 5. *Assume Conditions* (5) *and* (6) *of Main Theorem B, and let $h > 0$. There exist positive constants C, τ_0 with C depending on h and τ_0 not depending on h such that*

$$\int_\Sigma v(q, \tau_1)dV_q \le C\left(\frac{1}{\mu} + \inf_{p \in \Sigma} v(p, \tau_1 + h)\right)$$

for any $q \in \Sigma$ and $\tau_1 > \tau_0$.

Proof. Let $G(p, q)$ be the positive Green's function such that

$$v(p, \tau) = -\int \Delta v(q, \tau)G(p, q)dV_q + \frac{1}{\mathrm{Vol}_\gamma(\Sigma)}\int v(q, \tau)dV_q;$$

see [1]. Using (4) and rearranging, one has

$$\frac{1}{\mathrm{Vol}(\Sigma)}\int v(q, \tau)dV_q = \int \left(\frac{1}{v^2}\frac{\partial v}{\partial \tau} + \frac{1}{2}v - fv^3\right)G(p, q)dV_q + v(p, \tau)$$

$$\le -\int \left(\frac{\partial w}{\partial \tau} - \frac{1}{2}w\right)G(p, q)dV_q + v(p, \tau),$$

where $w = 1/v$. Multiplying by the integrating factor $e^{-\frac{\tau}{2}}$, this becomes

$$(14) \qquad e^{-\frac{\tau}{2}}\frac{1}{\mathrm{Vol}(\Sigma)}\int v(q, \tau)dV_q \le -\int \frac{\partial}{\partial \tau}\left(e^{-\frac{\tau}{2}}w\right)G(p, q)dV_q + e^{-\frac{\tau}{2}}v(p, \tau).$$

Assuming $\tau_1 > \tau_0$, $\tau \in (t_1, \tau_1 + h)$, and choosing τ large enough that $t > 1/4$, from (8) we have

$$\int v(q, \tau)dV_q \ge \frac{1}{2}e^{-\frac{(\tau - \tau_1)}{2}}\int v(q, \tau_1)dV_q \ge \frac{1}{2}e^{-\frac{(h)}{2}}\int v(q, \tau_1)dV_q$$

and similarly

$$v(p, \tau) \le 2e^{h/2}v(p, \tau_1 + h).$$

Using these two inequalities in (14) we get
(15)
$$e^{-\frac{\tau}{2}}\frac{1}{\mathrm{Vol}(\Sigma)}\int v(q, \tau_1)dV_q \le -2e^{h/2}\int \frac{\partial}{\partial \tau}\left(e^{-\frac{\tau}{2}}w\right)G(p, q)dV_q + e^{-\frac{\tau}{2}}4e^h v(p, \tau_1 + h).$$

Integrating over $(\tau_1, \tau_1 + h)$ yields

$$\frac{e^{\frac{\tau_1}{2}} - e^{\frac{\tau_1+h}{2}}}{2} \frac{1}{\text{Vol}(\Sigma)} \int v(q, \tau_1) dV_q$$

$$\leq 2e^{h/2} \int \left(w(q, \tau_1) e^{\frac{\tau_1}{2}} - w(q, \tau_1 + h) e^{\frac{\tau_1+h}{2}} \right) G(p, q) dV_q$$

$$+ \frac{e^{\frac{\tau_1}{2}} - e^{\frac{\tau_1+h}{2}}}{2} 4e^h v(p, \tau_1 + h).$$

Using now that $w \leq 1/\mu$, we obtain from this, finally

$$\frac{1}{\text{Vol}(\Sigma)} \int v(q, \tau_1) dV_q \leq \frac{4e^{\frac{h}{2}}}{\mu \left(1 - e^{\frac{h}{2}} \right)} \int G(p, q) dV_q + 4e^h v(p, \tau_1 + h).$$

$\square$

Lemma 6. *Let v be a solution of Equation (4) on $[1, \infty)$ satisfying*

$$v \geq \mu.$$

Fix $\tau \in [1, \infty)$ and let $M = \sup_\Sigma v(q, \tau)$, $f^* = \sup_\Sigma f(q, \tau)$. Then given $\varepsilon > 0$ there exists $r_0 > 0$ such that for $r < r_0$ there holds

$$(16) \qquad v(p, \tau) < \frac{r^2}{2} \left(\frac{1}{2\mu} \frac{e^{-1}}{1 - e^{-1}} + f^* M \right) + \frac{1 + \varepsilon}{|B_r(p)|} \int_{B_{r(p)}} v$$

for any $p \in \Sigma$.

Proof. For

$$H_0 = \frac{1}{2} \left(\frac{1}{2\mu} \frac{e^{-1}}{1 - e^{-1}} + f^* M \right)$$

define $\phi = v + H_0 r^2$. Now, on a neighborhood of p let $(r, \theta^1, \theta^2, ..., \theta^{n-2})$ be geodesic polar coordinates for γ at p, for which we may write $\gamma = dr^2 + r^2 h_{AB} d\theta^A d\theta^B$. Then (see [1] p. 20) for a number b such that b^2 bounds the sectional curvature of (Σ, γ) from above one has

$$\Delta r^2 = 2(n-1) + 2r\partial_r \log |h| \geq 2(n-1) + 2r\frac{\partial}{\partial r} \log \frac{\sin br}{r}.$$

And so, given $\delta > 0$ we may choose r_0 small enough such that $\Delta r^2 \geq 2(n-1) - \delta$ whenever $r < r_0$. Using now Equation (4), we have

$$\Delta \phi = \frac{1}{v^2} \frac{\partial v}{\partial \tau} + \frac{1}{2v} - fv + (2(n-1) - \delta) H_0.$$

Hence by (8)

$$\Delta \phi > -\frac{1}{2v(1 - e^{-\tau})} + \frac{1}{2v} - fv + (2(n-1) - \delta) H_0$$

$$= -\frac{1}{2v} \frac{e^{-\tau}}{1 - e^{-\tau}} - fv + (2(n-1) - \delta) H_0$$

$$\geq -\frac{1}{2\mu} \frac{e^{-1}}{1 - e^{-1}} - f^* M + (2(n-1) - \delta) H_0 > 0$$

for $\delta < 1$; ϕ is subharmonic, and the lemma follows from the mean value inequality for subharmonic functions by choosing r_0, perhaps, smaller still. $\square$

We are now in a position to bound $\sup v$ in terms of an integral of v.

Lemma 7. *Let v be a solution of Equation* (4) *on $[1, \infty)$ satisfying $v \geq \mu$. Given $\varepsilon > 0$ there is an $r < \sqrt{2/f^*}$ such that*

$$\sup_{p \in \Sigma} v(p, \tau) \leq \frac{2}{2 - r^2 f^*} \left(\frac{r^2}{4\mu} \frac{e^{-1}}{1 - e^{-1}} + \frac{1 + \varepsilon}{|B_r|} \int_{B_{r(p)}} v \right).$$

Proof. For $\tau \in [1, \infty)$ fixed define $M = \sup_\Sigma v(p, \tau)$ and let p be such that $v(p, \tau) = M$. Then by Lemma 6

$$M < \frac{r^2}{2} \left(\frac{1}{2\mu} \frac{e^{-1}}{1 - e^{-1}} + f^* M \right) + \frac{1 + \varepsilon}{|B_r|} \int_{B_{r(p)}} v.$$

Hence, for $r < \sqrt{2/f^*}$ there holds

$$M \leq \frac{2}{2 - r^2 f^*} \left(\frac{r^2}{4\mu} \frac{e^{-1}}{1 - e^{-1}} + \frac{1 + \varepsilon}{|B_r|} \int_{B_{r(p)}} v \right).$$

$\square$

Proposition 8. *Let v be a solution of Equation* (4) *on $[1, \infty)$ satisfying $v \geq \mu$ and let $h > 0$. There exists C independent of τ but depending on h such that*

$$\sup_{p \in \Sigma} v(p, \tau) \leq C \left(\frac{1}{\mu} + \inf_{p \in \Sigma} v(p, \tau + h) \right).$$

Proof. Apply Lemmata 5 and 7. $\square$

Theorem 1 can now be proved.

proof of Theorem 1. As previously remarked, the result follows directly from Proposition 8 if we can show that $\tilde{v}(\tau) = \inf_\Sigma v(p, \tau)$ remains bounded for all time. In fact, there holds $\tilde{v} < 1/\sqrt{2 \inf f}$. To see this, suppose instead that there is a time τ_1 at which $v(\tau_1) = 1/\sqrt{2 \inf f}$. Then v_* satisfying

$$v_*' = \inf f v_*^3 + \inf f v^3 - \frac{v_*}{2}$$

$$v_*(\tau_1) = \frac{1}{\sqrt{2 \inf f}}$$

is a subsolution of Equation (4) in the sense that

$$\frac{\partial v_*}{\partial \tau} \leq v_*^2 \Delta v_* + f v_*^3 - \frac{v_*}{2}.$$

The parabolic maximum principle shows that $v \geq v_*$. But v_* blows up in finite time, and thus v must also, which is a contradiction to the definition of v. $\square$

Finally, before leaving this section, we remark that any solution v of Equation (4) satisfying the bounds of Theorem 1 will in fact be uniformly bounded in C^k. That is, for every τ (large enough, of course) one will have $\|v(\tau)\|_{C^k(\Sigma)} \leq C$ for some constant only depending on μ, M. The crucial step towards doing this is to observe that on any finite interval of the form $I = [0, T]$ such a solution v will be uniformly bounded in the parabolic analogue H_I^α of C^α; for the precise definition of $H_I^{k,\alpha}$ see [15]. This Hölder continuity follows from estimates originally due to Moser;

see [9] Theorem 6.28 [1]. Afterwards, we may repeatedly apply standard parabolic Schauder theory to get that $\|v\|_{H_I^{k,\alpha}} \leq C$. The desired bounds follow since, given $\tau \in I$, one has $\|v(\tau)\|_{C^{k,\alpha}(\Sigma)} \leq \|v\|_{H^{k,\alpha}(\Sigma)}$.

4. Proof of Theorem 2

The main ingredient used in the proof of Theorem 2 is the functional

$$J(v) = \int_\Sigma |\nabla v|^2 - fv^2 + \log v,$$

which is easily seen to be non-increasing in τ by virtue of Condition (6) and Equation (4):

$$(17) \qquad \frac{\partial J}{\partial \tau} = -2 \int_\Sigma \frac{\partial v}{\partial \tau}\left(\Delta v + fv - \frac{1}{2v}\right) - \int_\Sigma \frac{\partial f}{\partial \tau} v^2 \leq -2\int_\Sigma \frac{1}{v^2}\left|\frac{\partial v}{\partial \tau}\right|^2$$

The hypothesis $\mu \leq v \leq M$ then establishes

$$(18) \qquad\qquad\qquad J(v) \geq -\sup f M^2 + \log \mu$$

$$(19) \qquad\qquad\qquad \int_\Sigma |\nabla v|^2 \leq J(v_0) + \sup M^2 - \log \mu.$$

From the latter, we immediately obtain:

Proposition 9. *Assuming the hypotheses of Theorem 2, there exists a sequence τ_i such that $v(\tau_i) \to \omega$ weakly in H^1 and strongly in L^1.*

Proof. Following the argument of the introduction to this section, the bound (19) shows that v is bounded in H^1. The result now follows from Rellich's theorem. In the case $n = 3$, in which Σ is 2-dimensional, one may apply Rellich's theorem to v as a function of $\Sigma \times \mathbb{S}$, for instance. $\square$

We are now in a position to prove Theorem 2, for which it only remains to be shown that ω is a stationary state and the convergence is actually in C^k for any k.

proof of theorem 2. Let $v(\tau_i)$ be as in the conclusion of the preceding proposition, fix $T > 0$, and let $h < T$. Using the bound $v \leq M$ and (17), we get

$$\frac{1}{M^2}\int_\Sigma \left|\frac{\partial v}{\partial \tau}\right|^2 \leq -\frac{1}{2}\frac{d}{d\tau}J(v).$$

Integrating over $[\tau_i, \tau_i + h]$ and two applications of Hölder's inequality yields

$$\|v(\tau_i + h) - v(\tau_i)\|_{L^1} \leq C\sqrt{h}\sqrt{J(v(\tau_i)) - J(v(\tau_i + h))},$$

for $C \geq \mathrm{Vol}_\gamma(\Sigma)$. Since we know that the right hand side converges, we get that $v(\tau_i + h) \to w$ in L^1 uniformly in h.

Now since $\mu \leq v \leq M$ it follows that $\mu \leq \omega \leq M$ a.e.; whence

$$\int_\Sigma \left|\frac{1}{v(\tau_i)} - \frac{1}{\omega}\right| \leq \frac{1}{\mu^2}\int_\Sigma |\omega - v(\tau_i)|,$$

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[1] In order to obtain the Hölder regularity using Moser's techniques one must render the equation into divergence form. In our case this gives gradient squared terms that threaten to violate the necessary structure conditions of, for instance, [9]. However, careful checking shows that they do not. The reason for this is that in the case that we have good L^∞ bounds on u, Moser's techniques can be implemented without putting the equation into divergence form, thereby avoiding the introduction of these dangerous gradient terms in the first place; see [14]

and so $v^{-1}(\tau_i + h) \to \omega^{-1}$ in L^1 as well.

We are now in a position to prove that the limiting function ω is a solution of the stationary equation. Let ψ be a C^∞ function on Σ, and $\varphi(\tau)$ a C^∞ function compactly supported on $[0, T]$, and put $v_i(p, \tau) = v(p, \tau_i + \tau)$. Then

$$\int_0^T \int_\Sigma \frac{\psi\varphi'}{v_i} = \int_0^T \int_\Sigma -\varphi\nabla\psi \cdot \nabla v_i + f\varphi\psi v_i - \frac{\varphi\psi}{v_i},$$

and the convergence results from the previous paragraph show that

$$0 = \int_0^T \int_\Sigma -\varphi\nabla\psi \cdot \nabla\omega + f\varphi\psi\omega - \frac{\varphi\psi}{\omega}.$$

Hence ω is a weak solution of the stationary equation that is essentially bounded above and below by positive constants. The fact that it is a C^∞ solution follows from the Sobolev embedding theorem and elliptic regularity.

To complete the proof, it only remains to show that the convergence may also be taken in the sense of C^k. To do so, we look at the differences $\delta v_i = v_i - \omega$, which verify

$$\delta v_i' = v_i^2 \Delta \delta v_i + \left(f v_i^2 + \frac{1}{2v_i\omega}\right)\delta v_i + (f_i - f)v_i,$$

where we have put $f_i(\tau) = f(\tau + \tau_i)$. Note that by the remarks at the end of Section 2, the v_i are uniformly bounded in H_I^α for finite intervals $I = [0, T]$. Whence, we may regard the previous equations as linear equations with uniformly Hölder continuous coefficients. By the standard parabolic regularity theory we may conclude

$$\|\delta v_i\|_{L^{2,q}(\Sigma \times \tilde{I})} \le C\left(\|\delta v_i\|_{L^q(\Sigma \times I)} + \|f_i - f\|_{L^q(\Sigma \times I)}\right),$$

for any $q > 1$ and $\tilde{I} = [h_0, T], h_0 > 0$; see [9] p. 172. But since the v_i are uniformly bounded, convergence of δv_i to 0 in L^1 implies convergence in L^q, whereupon the previous inequality yields $L^{2,q}$ convergence, and the Sobolev embedding theorem implies H^α convergence for some $0 < \alpha < 1$. This implies $H^{k,\alpha}$ convergence since by the parabolic Schauder theory we have a bound of the form

$$\|\delta v_i\|_{H_{\tilde{I}}^{k,\alpha}} \le C\left(\|\delta v_i\|_{C^0(\Sigma \times I)} + \|f_i - f\|_{H_I^\alpha}\right).$$

The theorem follows.   □

5. Proof of Theorem 3

In this section it is proved that the stationary state ω found in the last section as the limit of certain sequences $v(\tau_i)$ is not only unique, but in fact $\lim_{\tau \to \infty} v(\tau) = \omega$, where the limit may be taken in the sense of C^k. This will be proved by using a result of Leon Simon, which will imply in our case that for τ_1 sufficiently large and $\tau > \tau_1$ there holds

$$(20) \qquad c\|\log v(\tau_1) - \log v(\tau)\|_{L^2} \le \int_{\tau_1}^\infty \left\|(\log u)'\right\|_{L^2} \le |J(v(\tau_1)) - J(w)|^\theta$$

for some $\theta \in (0, 1/2)$. From this it is obvious that $\lim_{\tau \to \infty} \log v \to \log \omega$ in the sense of L^2, and convergence in C^k is then a matter of standard regularity theory.

To describe Simon's result, consider the general case of a parabolic equation

$$(21) \qquad\qquad\qquad\qquad \nu' = \mathcal{M}(\nu)$$

that arises from an elliptic differential operator

$$\mathcal{M} : C^{k,\alpha} \cap L^2 \to C^{k-2,\alpha} \cap L^2,$$

for which we assume that there exists an energy functional $\mathcal{E} : L^2 \to \mathbb{R}$ such that

$$< \mathcal{M}(\nu), \xi >= -\frac{d}{ds}\mathcal{E}(\nu + s\xi)|_{s=0}$$

for any $\xi, \nu \in C^{k,\alpha} \cap L^2$. Assume furthermore that

$$\mathcal{E}(\nu) = \int E(q, \nu, \nabla\nu)$$

for some smooth $E : M \times \mathbb{R} \times T_pM \to \mathbb{R}$. Finally, we assume E to be analytic, uniformly in q, as a function on $\mathbb{R} \times T_qM$ in the following sense:

$$(22) \qquad E(q, s_1\nu, s_2\chi) = \Sigma_{|\alpha|\geq 0}E_\alpha(q, \nu, \chi)s^\alpha, \ s = (s_1, s_2)$$

The following result is found in [12]:

Theorem 10. *Let $\mathcal{M}, \mathcal{E}, E$ be as above, where E satisfies the analyticity hypothesis (22). Assume furthermore that $\mathcal{M}(0) = 0$ and the linearization L of $\mathcal{M}$ at 0 is uniformly elliptic. Then there exists σ such that for $\|\nu\|_{C^{2,\alpha}} < \sigma$ there holds, for some constant c,*

$$(23) \qquad \|\mathcal{M}(\nu)\|_{L^2} \geq c|\mathcal{E}(\nu) - \mathcal{E}(0)|^{1-\theta}$$

for some $\theta \in (0, 1/2)$.

Remark 11. In [12] the convexity condition $E(x, z, p) \geq c|p|^2$ is assumed. This condition does not appear directly in the proof of (23), but is used to infer the uniform ellipticity of L. Since we assume the latter outright we may dispense with the convexity assumption.

As in [12], this can be used to prove

Corollary 12. *Let σ, θ be as in the previous theorem. Suppose that $\mathcal{E}(\tau, \nu)$ is non-increasing for ν a solution of Equation (21) and also $\mathcal{E}(\tau, \phi)$ is non-increasing for fixed $\phi \in C^\infty(\Sigma)$. Then if ν is a solution of Equation (21) satisfying $\|\nu(\tau)\|_{C^{2,\alpha}} < \sigma$ for $\tau \in (\tau_1, \tau_2)$ there holds*

$$(24) \qquad c\int_{\tau_1}^{\tau_2} \|\nu'\|_{L^2} \leq \frac{1}{\theta}|\mathcal{E}(\nu(\tau_1)) - \mathcal{E}(0)|^\theta.$$

Proof. Denote by $\mathcal{E}'(\nu)$ the partial of $\mathcal{E}$ with respect to its intrinsic time dependence: $\mathcal{E}'(\tau_0, \nu(\tau_0)) = \frac{d}{d\tau}\mathcal{E}(\tau, v(\tau_0))|_{\tau_0}$ Using the fact that $\mathcal{E}$ is the energy functional for $\mathcal{M}(\nu)$, the evolution equation $\nu' = \mathcal{M}(\nu)$, and one of the monotonicity assumptions for $\mathcal{E}$ one has

$$-\frac{d}{d\tau}\mathcal{E}(\nu) =< \mathcal{M}(\nu), \nu' > -\mathcal{E}'(\nu) = \|M(\nu)\|_{L^2} \|\nu'\|_{L^2} - \mathcal{E}'(\nu)$$

$$\geq \|M(\nu)\|_{L^2} \|\nu'\|_{L^2} .$$

Applying the previous theorem, for $\tau \in (\tau_1, \tau_2)$ there holds

$$(25) \qquad -\frac{1}{(\mathcal{E}(\nu) - \mathcal{E}(0))^{1-\theta}} \frac{d}{d\tau}\mathcal{E}(\nu) \geq c \|\nu'\|_{L^2} .$$

The Corollary is now obtained by integration. $\qquad\qquad\qquad\qquad\square$

Theorem 3 can now be proved.

Proof of Theorem 3. By the hypothesis to this theorem, given $\varepsilon > 0$, there exists τ_1, which may be chosen arbitrarily large, such that $|v(\tau_1) - \omega| < \varepsilon$. By using the maximum principle and the bounds on v it can be arranged that $|v(\tau) - \omega| < \varepsilon$ for $\tau \in [\tau_1, \tau_1 + T]$, T fixed. Defining $\tilde{\omega} = \log \omega$ and $\nu = \log(v/\omega)$, this will be used below to assume that $\|\nu\|_{C^0}$ is as small as needed on an interval $[\tau_1, \tau_1 + T]$.

Define

$$\mathcal{M}(\nu) = e^{(\tilde{\omega}+\nu)} \left(\Delta e^{(\tilde{\omega}+\nu)} + f e^{(\tilde{\omega}+\nu)} \right) - \frac{1}{2},$$

and note that Equation (4) can now be written in the form (21). Since ω is a stationary state of (4) it follows that $\mathcal{M}(0) = 0$. Define

$$E(q, z, p) = e^{2(\tilde{\omega}+z)} \left((p + \nabla\tilde{\omega})^2 - f \right) - \frac{1}{2}\tilde{v}.$$

Then since $\log \mu \leq \tilde{\omega} \leq \log M$, the function E is uniformly analytic in the sense of (22). Note that $\mathcal{E}(\nu) \equiv \int E(q, \nu, \nabla\nu) dV_q$ is an energy functional for $\mathcal{M}$. In fact $\mathcal{E}(\nu) = J(v)$, and from this it follows that $\mathcal{E}(\tau) = \mathcal{E}(\nu(\tau))$ is non-increasing and $\lim_{\tau\to\infty} \mathcal{E}(\tau) = \mathcal{E}(0)$. The other monotonicity condition that we require comes directly from the fact that $f' \geq 0$. Thus, we are almost in a position to apply the previous theorem; all that remains is to show that ν eventually does not leave a small $C^{2,\alpha}$ neighborhood of $\tilde{\omega}$. To do so, we shall apply (25) over small time intervals, but since this estimate is an L^2 estimate we need to use a little regularity theory to convert this to a $C^{2,\alpha}$ estimate.

To obtain the required estimates, we must use the fact that $\omega = e^{\tilde{\omega}}$ is a solution of the stationary equation to write the equation for ν as

$$\nu' = \nabla \cdot \left(e^{2(\tilde{\omega}+\nu)} \nabla\nu \right) - \left(e^{2(\tilde{\omega}+\nu)} \nabla\nu \right) \cdot \nabla\nu + \left(\frac{e^{2\nu} - 1}{2\nu} \right) \nu.$$

Using now the remarks at the end of Section 2 we have that v, ω are $H^{1,\alpha}$ on finite time intervals, and thus we may view the previous equation as a linear equation for ν with Hölder continuous coefficients so that standard parabolic regularity theory yields:

$$(26) \qquad \sup_{[\tau_0+h, \tau_1]} \|\nu(\tau)\|_{C^{k,\alpha}} \leq C \left(\sup_{[\tau_0, \tau_1]} \|\nu(\tau)\|_{L^2} + \sup_{\tau_0} |\nu| \right),$$

for h on some fixed interval $[h_0, h_1] \in \mathbb{R}^+$, any k, and some $C = C(h)$ for which we may, of course, assume $C > 1$.

Let σ, θ be as in Corollary 12. Using the remarks of the first paragraph, the bound (26), and the convergence of $\mathcal{E}(\tau)$ we can choose τ_1 such that $\|\nu(\tau)\|_{C^{2,\alpha}} < \sigma$, for $\tau \in [\tau_1, \tau_1 + h_0]$, $\|\tilde{v}(\tau_1)\|_{L^2} < \frac{\sigma}{6C}$, and for any $\tau > \tau_1$ we have

$$\frac{1}{\theta} |\mathcal{E}(\nu(\tau)) - \mathcal{E}(0)| < \frac{\sigma}{4C}.$$

Claim: for $\tau > \tau_1$ there holds $\|\nu\|_{C^{2,\alpha}} < \sigma$ so that the bound (24) applies on (τ_1, ∞). Indeed, suppose otherwise. Then there is a maximal τ, say τ_2, for which $\|\nu\|_{C^{2,\alpha}} < \sigma$ for $\tau \in [\tau_1, \tau_2)$ but $\|\nu(\tau_2)\|_{C^{2,\alpha}} = \sigma$. However, the bounds (24) and (26) together with the assumption on τ_1 show that

$$\|\nu(\tau_2)\|_{C^{2,\alpha}} \leq C \|\nu(\tau_2 - h_0)\|_{L^2} \leq C \left(\frac{1}{\theta} |\mathcal{E}(\tilde{v}(\tau)) - \mathcal{E}(0)| + \|\nu(\tau_1)\|_{L^2} \right) \leq \frac{\sigma}{2},$$

which is a contradiction. Thus, we may apply (24) on (τ_1, ∞) to get, for any $\tau_3, \tau_4 > \tau_1$

$$\|\nu_{\tau_3} - \nu_{\tau_4}\| \leq \frac{1}{\theta} |\mathcal{E}(\nu(\tau_3)) - \mathcal{E}(\nu(\tau_4))|.$$

Since R.H.S $\to 0$ as $\tau_3, \tau_4 \to \infty$, it follows that $\nu(\tau)$ converges in L^2 to a unique limit, which must be $\tilde{\omega}$. The convergence in $C^{k,\alpha}$ follows from (26). $\square$

6. STABILITY OF THE SPHERICALLY SYMMETRIC BLOW-UP

In order to prove the stability of the spherically symmetric blow-up, we shall continue to deal with Equation (2). Thus, equivalently, we are to prove:

Theorem 13. *Let (Σ, γ) be the fixed round $(n-1)$-sphere, and assume $f \equiv 1/2$. There exists a neighborhood B of 1 in $C^{2,\alpha}(\mathbb{S}^{n-1})$ and a Lipschitz function $\tilde{r} : B \to \mathbb{R}$ such that given $u_0 \in B$, the solution of Equation (2) with initial data u_0 at $\tau = 0$ is of the form*

$$u = \frac{v}{\sqrt{1 - \frac{t}{T}}},$$

where v is bounded and $\lim_{t \to T} v = 1$.

This theorem, in turn, is proven by demonstrating the asymptotic stability of the 0 solution for

$$(27) \qquad \frac{\partial \nu}{\partial \tau} = e^\nu \Delta e^\nu + \frac{e^{2\nu} - 1}{2\nu} \nu,$$

where $\nu = \log v$ and τ is as before. The arguments used will involve both the maximum principle and integral techniques, and so we shall first of all require a technical theorem for converting L^2 bounds into $C^{2,\alpha}$ bounds.

Proposition 14. *Given $\varepsilon > 0$ there exists a $\delta > 0$ such that given a solution ν of Equation (27) on an interval $I = [0, T]$ satisfying $\|\nu(0, \cdot)\|_{C^{2,\alpha+\beta}}, \sup_I \|\nu(t, \cdot)\|_{L^2} \leq \delta$, one has $\sup_I \|\nu\|_{C^{2,\alpha}} < \varepsilon$.*

Proof. Let $H^{k,\alpha}$ denote the standard parabolic Hölder space as discussed earlier, and for some interval $\tilde{I} = [0, \tilde{T}]$ let $M > 0$ be such that $M \geq \|u\|_{H_{\tilde{I}}^{1,\alpha}}$. By using the parabolic Schauder theory and weak parabolic Harnack inequalities we have that there exists $C = C(M)$ such that

$$\|\nu\|_{H_{\tilde{I}}^{2,\alpha}} \leq C(M) \left(\|\nu(0, \cdot)\|_{C^{2,\alpha}} + \sup_{\tilde{I}} \|\nu\|_{L^2} \right).$$

We take $\delta = \min\{\varepsilon/4C(2\varepsilon), \varepsilon\}$. To see that the theorem is true with this choice of δ, suppose it were not and let $\tilde{I}$ be the maximal subinterval for which $|\nu|_{H_{\tilde{I}}^{k,\alpha}} < \varepsilon$. Then by the compactness of $H^{1,\alpha}$ in $H^{2,\alpha}$ we still have $\|\nu\|_{H_{\tilde{I}}^{1,\alpha}} < 2\varepsilon$ on this interval, and so the previous estimate gives that in fact $\|\nu\|_{H_{\tilde{I}}^{k,\alpha}} < \varepsilon/2$, which is a contradiction. $\square$

The central result that we shall need is that on small neighborhoods the 0 average part of ν is contracting. That is, defining $\bar{\nu} = \frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \nu$ and $\tilde{\nu} = \nu - \bar{\nu}$, one has the following:

Proposition 15. *Let $0 < \sigma < 1$. There exists $\varepsilon > 0$ such that if ν is not identically 0 and $|\bar{\nu}|, |\tilde{\nu}| < \varepsilon$ for $\tau > \tau_0$ then $\|\tilde{\nu}\|_{L^2} < e^{-(1-\sigma)\tau} \|\tilde{\nu}(\tau_0, \cdot)\|_{L^2}$.*

Proof. First it should be noted that $\tilde{\nu}$ is not identically 0. To see this, suppose that it were. Then $\bar{\nu}$ would have to satisfy

$$\frac{d\bar{\nu}}{d\tau} = \frac{e^{2\bar{\nu}} - 1}{2}.$$

But then either $\bar{\nu}$ would blow up in finite time or $\bar{\nu} = 0$, both of which contradict the hypotheses of the theorem.

To continue, we multiply Equation (27) by $\tilde{\nu}$ and integrate over $\mathbb{S}^{n-1}$ to get

$$(28) \qquad \frac{1}{2}\int (\tilde{\nu}^2)' = \int -e^{2\nu}(1+\tilde{\nu})|\nabla\tilde{\nu}|^2 + \frac{\tilde{\nu}}{2}\left(e^{2\nu} - 1\right).$$

But since the integral of $\tilde{\nu}$ vanishes (by definition), we have for the last term:

$$\int \tilde{\nu}\left(\frac{e^{2\nu}-1}{2}\right) = \int \tilde{\nu}\frac{e^{2\nu}}{2} = \int \tilde{\nu}\frac{e^{2\bar{\nu}}e^{2\tilde{\nu}}}{2} = \int \tilde{\nu}e^{2\bar{\nu}}\left(\frac{e^{2\tilde{\nu}}-1}{2}\right)$$

Substituting this in (6) and replacing $\tilde{\nu}$ with $|\tilde{\nu}|$ in the gradient term yields the following inequality:

$$\frac{1}{2}\int(\tilde{\nu}^2)' \le \int -e^{2\nu}(1+|\tilde{\nu}|)|\nabla\tilde{\nu}|^2 + \tilde{\nu}^2 e^{2\bar{\nu}}\left(\frac{e^{2\tilde{\nu}}-1}{2\tilde{\nu}}\right).$$

Using the hypothesis on the size of ν gives

$$\frac{1}{2}\int(\tilde{\nu}^2)' \le \int -e^{4\varepsilon}(1-\varepsilon)|\nabla\tilde{\nu}|^2 + e^{2\varepsilon}\frac{e^{2\varepsilon}-1}{2\varepsilon}\tilde{\nu}^2.$$

But by the Poincaré inequality $\|\nabla\tilde{\nu}\|_2^2 \ge \lambda_1\|\tilde{\nu}\|_2^2$, where $\lambda_1 \ge 2$ is the first nonzero eigenvalue of the Laplacian on $\mathbb{S}^{n-1}$, so that we have finally

$$\frac{1}{2}\frac{d}{d\tau}\|\tilde{\nu}\|_2^2 \le -\left(\lambda_1 e^{4\varepsilon}(1-\varepsilon) - e^{2\nu}\frac{e^{2\nu}-1}{2\varepsilon}\right)\|\tilde{\nu}\|_2^2.$$

Choosing ε so that the term in parentheses is greater than $1-\sigma$, integration against $e^{2(1-\sigma)\tau}$ yields $\|\tilde{\nu}\|_2^2 < \|\tilde{\nu}_0\|_2^2\, e^{-(1-\sigma)\tau}$ since we know that $\tilde{\nu}$ is not identically 0. Whence $\|\tilde{\nu}\|_2 < \|\tilde{\nu}_0\|_2\, e^{-\frac{(1-\sigma)}{2}\tau}$. $\qquad\square$

As a corollary, we get the uniqueness of the equilibrium solution $\nu \equiv 0$ on a small neighborhood thereof.

Corollary 16. *Let ν be a function on Σ that satisfies $|\tilde{\nu}|, |\bar{\nu}| < \varepsilon$ and*

$$(29) \qquad\qquad 0 = e^\nu\Delta e^\nu + \frac{1}{2}e^{2\nu} - \frac{1}{2}.$$

Then in fact $\nu \equiv 0$.

Proof. Taking $\nu(\tau, p) = \nu(p)$, ν certainly satisfies the evolution equation and so by the previous theorem if ν is not identically 0 then $\|\tilde{\nu}\|_2 < \|\tilde{\nu}\|_2\, e^{-(1-\delta)\tau}$, which is a contradiction. Hence $\nu = \bar{\nu}$; that is, ν is a constant. But then (29) implies that $e^{2\nu} = 1$, which implies the result. $\qquad\square$

But although the 0 average part may be contracting, the maximum principle shows that for most choices of $\bar{\nu}$, the solution will in fact leave a small neighborhood:

Proposition 17. *Suppose that $|\tilde{\nu}(0, \cdot)| < \delta$ then there exist nonempty disjoint open sets $A, B \subset (-\delta, \delta)$ such that if $\bar{\nu} \in A$ initially then $\bar{\nu} > \delta$ eventually and if $\bar{\nu} \in B$ initially then $\bar{\nu} < -\delta$ eventually.*

Proof. In fact, we can choose $A = (|\tilde{\nu}(0, \cdot)|_0, \delta)$ and $B = (-\delta, -|\tilde{\nu}(0, \cdot)|)$. For if $\bar{\nu} \in A$ we have that $\nu > 0$ initially and hence by the maximum principle is bounded below by the solution ν^* of

$$\frac{d\nu^*}{d\tau} = e^{\nu^*} - 1$$
$$\nu^* = \nu(0, \cdot).$$

Solutions of the latter certainly become greater than δ in finite time. Since $\nu = \tilde{\nu} + \bar{\nu}$ and $\tilde{\nu}$ is 0 somewhere for every τ it follows that $\bar{\nu}$ must become greater than δ as well. The proof of the statement for B is exactly the same. $\qquad\square$

To use the information of the previous theorem, it is useful to define the cylinders Q_δ and Q'_δ as follows:

$$Q_\delta = \{\nu : \|\tilde{\nu}\|_2, |\bar{\nu}| < \delta\}$$
$$Q'_\delta = \{\nu : |\tilde{\nu}|, |\bar{\nu}| < \delta\}$$

In both of these cases we refer to the sets where $\bar{\nu} = +(-)\delta$ as the top(bottom) of the cylinder. We are now able to apply Theorems 15 and 17 to solutions with initial data in these cylinders to obtain our first stability result.

Proposition 18. *There exists δ such that for $\tilde{\nu}$ satisfying $\int_{\mathbb{S}^{n-1}} \tilde{\nu} = 0, |\tilde{\nu}| < \delta$ there exists $\bar{\nu}$ with $|\bar{\nu}| < \delta$ such that the solution ν with initial data $\nu = \tilde{\nu} + \bar{\nu}$ exists and remains in Q_δ for all later time. Furthermore $\lim_{\tau \to \infty} \nu = 0$.*

Proof. To choose δ, let ε be as in Theorem 15. With $C(\varepsilon)$ as in Theorem 14, take $\delta = \varepsilon/(2C(\varepsilon))$. Let $\lambda \in (-\delta, \delta)$ and let v_λ be the solution of Equation (27) with initial data $\tilde{\nu} + \lambda$. Note that initially $\nu_\lambda \in Q'_\delta \subset Q_\delta$ and by Theorem 14 we have that $|\nu| < \varepsilon$ for as long as ν_λ remains in Q_λ. But then theorem 15 applies to show that if v_λ leaves Q_δ it must do so through the top or the bottom. By Theorem 17 the set of λ for which this happens forms nonempty disjoint open subsets of $(-\delta, \delta)$. By the connectedness of this interval, there exists a λ for which v_λ neither leaves though the top nor the bottom and thus remains in Q_λ for as long as it exists, which is forever since $|\nu| < \varepsilon$. By the results of Section 5, v_λ must converge to an equilibrium solution, which must be 0 by Theorem 16. $\qquad\square$

Another way to phrase the assertion of the previous theorem is that there exists a function $\bar{\nu} = \nu(\tilde{\nu})$ defined on a neighborhood of the origin in the space of L^2 functions of average 0 such that there exists a solution of Equation (27) with initial data $\tilde{\nu} + \bar{\nu}(\tilde{\nu})$ that converges to 0 as $\tau \to \infty$. The next theorem asserts that this function is unique and Lipschitz continuous.

Proposition 19. *Let $\bar{\nu} = \bar{\nu}(\tilde{\nu})$ be the function in the preceding remarks. Then*

$$|\bar{\nu}(\tilde{\nu}_1) - \bar{\nu}(\tilde{\nu}_0)| \le \|\tilde{\nu}_1 - \tilde{\nu}_0\|_{C_0}.$$

Proof. For convenience define $\delta\tilde{\nu} = \tilde{\nu}_1 - \tilde{\nu}_0$. Then we claim that

$$\bar{\nu}(\tilde{\nu}_0 + \delta\tilde{\nu}) - \bar{\nu}(\tilde{\nu}_0) + \delta\tilde{\nu}$$

cannot be either strictly positive or negative.

To prove the claim of the previous paragraph suppose that the above expression were either strictly positive or strictly negative. Then for $\nu_0 = \bar{\nu}(\tilde{\nu}_0) + \tilde{\nu}_0$ and $\nu_1 = \bar{\nu}(\tilde{\nu}_1) + \tilde{\nu}_1$ we have $\nu_1 - \nu_0$ is either strictly positive or strictly negative.

Extending these as functions of τ also so that they are solutions of Equation (27) we have by the maximum principle that ν_1 is either strictly greater or strictly less than ν_0. More precisely, defining $w_i = e^{-2\nu_i}$, $i = 1, 2$ and $\delta w = w_1 - w_0$ we have that δw verifies

$$(\delta w)' = \Delta \left(\frac{\delta w}{w_1 w_2} \right) + \left(\frac{f}{w_1 w_2} + \frac{1}{2} \right) \delta w.$$

We use the maximum principle to find that if δw is either strictly positive or strictly negative at $\tau = 0$ then $\delta w \geq \sup \delta w|_{\tau_0} e^{c\tau}$ or $\delta w \leq \inf \delta w|_{\tau_0} e^{c\tau}$, which is impossible since $\nu_i = \log w_i$ are supposed to be uniformly bounded for all time.

Thus, for some points $p, q \in \mathbb{S}^{n-1}$ we have that

$$- \|\delta\tilde{\nu}\|_{C_0} \leq -\delta\tilde{\nu}(p) < \bar{\nu}(\tilde{\nu}_0 + \delta\tilde{\nu}) - \bar{\nu}(\tilde{\nu}_0) < -\delta\tilde{\nu}(q) \leq \|\delta\tilde{\nu}\|_{C^0},$$

and this establishes the result.   □

Finally, we may prove our main stability result.

proof of Theorem4. Let $\nu_0 = \log u_0$, and let $\tilde{\nu} = \nu_0 - \frac{\int \nu_0}{\omega_n}$. Let $\bar{\nu}(\tilde{\nu})$ be as in the previous two theorems so that a bounded (above and below by positive constants) solution of Equation (27) exists for $\tau \to \infty$. This is equivalent to the existence of a solution v of Equation (2) with the re-scaled initial data $v(0, \cdot) = u_0 e^\alpha$ for $\alpha = \bar{\nu} - \frac{\int \nu_0}{\omega_n}$; using the previous theorem we see that α is Lipschitz continuous in u_0. For Equation (2) this yields a solution

$$\tilde{u} = \frac{e^\alpha v}{\sqrt{1-t}}$$

with initial data $e^\alpha u_0$. Recall however that solutions of Equation (2) scale according to $u(p, t) \to \lambda u(p, \lambda^2 t)$. Hence we also have the solution

$$\tilde{u} = \frac{v}{\sqrt{1 - e^{-2\alpha} t}},$$

and the theorem is proved taking $T = e^{2\alpha}$.   □

7. Blow-up solutions generated by the curve shortening flow

Recall that a closed, simply connected, parameterized curve in the plane $\gamma(\tau)$ is said to flow by curve shortening flow if

$$\frac{\partial \gamma}{\partial \tau} = k\vec{n},$$

where k is the curvature and $\vec{n}$ is the inward pointing unit normal. In the analysis it is often more convenient to use the support function and normal angle; given a point p on γ the normal angle is the angle between the position vector and the normal vector, and the support function is defined by $S(\theta) = \vec{n} \cdot \gamma$. Note that in the case that γ is convex the normal angle gives a paramterization of γ on $[0, 2\pi]$. The support function and curvature satisfy

$$S_t = -\frac{1}{S_{\theta\theta} + S}$$

and

$$k_t = k^2(k_{\theta\theta} + k).$$

Thus, in the case that the curve is strictly convex ($k > 0$), solutions of the curve shortening flow yield solutions of the parabolic scalar curvature equation on $\Sigma = \mathbb{S} \times \mathbb{S}$ with the product metric by taking $r^2 R - \kappa \equiv 1$ and $u(\theta_1, \theta_2) = \kappa(\theta_1)$.

Now, it is well known [8] that under the curve shortening flow, the curve will shrink to a point in finite time, and more specifically, the enclosed area $A(t)$ behaves according to $A(t) = A(0) - 2\pi t$. To study this shrinking more precisely one considers the normalized curve

$$\gamma(\cdot, t) = \left(\frac{\pi}{A(t)} \right)^{\frac{1}{2}} (\gamma(\cdot, t) - \gamma(\cdot, \omega)),$$

where $\omega = A(0)/(2\pi)$. The curvature of this normalized curve is given by

$$\tilde{k} = \left(\frac{A(t)}{\pi} \right)^{\frac{1}{2}} k.$$

Since it is well known from the work of Gage and Hamilton that the normalized curve converges to the round sphere as $t \to \omega$ [8], it follows that the normalized curvature converges to 1. Hence, u has the behavior claimed:

$$u = \frac{v}{\sqrt{A(0) - 2\pi t}},$$

where $v(\theta_1, \theta_2) = \tilde{k}(\theta_1)$ is uniformly bounded and in general can be assumed to vary in θ_1 simply by assuming that the starting curve is different from the circle.

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